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RELATION BETWEEN THE ZEROS OF A RATIONAL INTEGRAL FUNCTION AND ITS DERIVATE.

BY TSURUICHI HAYASHI.

The theorem: If $f'(z)$ is the first derivate of a rational integral function $f(z)$, then on the Gaussian plane of complex numbers all the roots of the equation $f'(z) = 0$ lie inside the smallest convex rectilinear polygon surrounding all the roots of the equation $f(z) = 0$, is well known.* All the proofs for this algebraical theorem are given by using dynamical conceptions, as far as I know.† So I shall prove it here without using any idea of force, but still by starting from the equation

$$(1) \qquad \frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} = 0,$$

in which $z_1, z_2, \cdots, z_n$ are the roots of $f(z) = 0$. Let a be any complex number and let equation (1) take the form

$$\sum_{r=1}^{n} \frac{1}{1 - \dfrac{z_r - a}{z - a}} = 0.$$

Put

$$z - a = \xi + \eta i, \qquad z_r - a = \xi_r + \eta_r i,$$

then

$$(2) \qquad \sum_{r=1}^{n} \frac{1}{1 - \dfrac{z_r - a}{z - a}} = \sum_{r=1}^{n} \frac{\left(1 - \dfrac{\xi\xi_r + \eta\eta_r}{\xi^2 + \eta^2}\right) + \dfrac{\xi\eta_r - \eta\xi_r}{\xi^2 + \eta^2} i}{\left(1 - \dfrac{\xi\xi_r + \eta\eta_r}{\xi^2 + \eta^2}\right)^2 + \left(\dfrac{\xi\eta_r - \eta\xi_r}{\xi^2 + \eta^2}\right)^2} = 0.$$

If we observe the imaginary part only of this equation, we find that ξ, η can not take those values for which all the expressions $\xi\eta_r - \eta\xi_r$ have the same sign,‡ since, if this be so, the imaginary part does not vanish. Now the equations

$$\xi\eta_r - \eta\xi_r = 0 \qquad (r = 1, 2, \cdots, n)$$

represent the straight lines joining the point a to the points $z_1, z_2, \cdots, z_n$.

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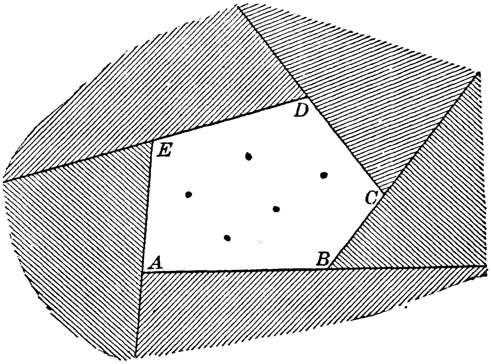
* For the literature of this theorem consult L. Fejér's paper "Ueber die Würzel vom kleinsten absoluten Betrage einer algebraischen Gleichung," Math. Annalen, Bd. 65, S. 417, 1908, E. Cesàros' paper, Nouv. Annales de Math., (3)4, p. 329, 1885, and Cesàro-Kowalewski's "Elementares Lehrbuch der algebraischen Analysis," S. 434, 1904.

† See L. Fejér's paper, W. F. Osgood's "Lehrbuch der Funktionentheorie," Bd. I, S. 176 1907, etc.

‡ Some of these expressions may be zero.

Hence if we make the point coincide with one of the vertices, A say, of the convex rectilinear polygon under consideration, then we find that the point (ξ, η), and therefore the root z of $f'(z) = 0$, can not lie within the exterior angles at A, since these regions are on the same-sign sides of all these straight lines. Next let the point a come to the vertex B; then the root of $f'(z) = 0$ can not lie within the exterior angles at B. Thus by letting the point a coincide with the vertices of the polygon successively, we find that all the roots of $f'(z) = 0$ can not lie within the shaded portion as in the accompaning diagram. If we consider the real part of equation (2), and the circles



$$1 - \frac{\xi\xi_r + \eta\eta_r}{\xi^2 + \eta^2} = 0$$

whose diameters are the joins of the points a and z_i, we can arrive at the same result.

SENDAI, JAPAN,
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